

SOME APPLICATIONS AND CONSTRUCTIONS OF INTERTWINING OPERATORS IN LCFT

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ABSTRACT. We discuss some applications of fusion rules and intertwining operators in the representation theory of cyclic orbifolds of the triplet vertex operator algebra. We prove that the classification of irreducible modules for the orbifold vertex algebra $\mathcal{W}(p)^{A_m}$ follows from a conjectural fusion rules formula for the singlet vertex algebra modules. In the $p = 2$ case, we computed fusion rules for the irreducible singlet vertex algebra modules by using intertwining operators. This result implies the classification of irreducible modules for $\mathcal{W}(2)^{A_m}$, conjectured previously in [4]. The main technical tool is a new deformed realization of the triplet and singlet vertex algebras, which is used to construct certain intertwining operators that can not be detected by using standard free field realizations.

1. INTRODUCTION

The orbifold theory with respect to a finite group of automorphisms is an important part of vertex algebra theory. There is a vast literature on this subject in the case of familiar rational vertex algebras, including lattice vertex algebras. It is widely believed that all irreducible representations of the orbifold algebra V^G (G is finite) arise from the ordinary and g -twisted V -modules via restriction to the subalgebra. This is known to hold if V^G is regular [14]. In fact, one reason for a great success of orbifold theory in the regular case comes from the underlying modular tensor category structure (e.g., quantum dimensions, Verlinde formula, etc.).

For irrational C_2 -cofinite vertex algebra this theory is much less developed and only a few known examples have been studied, such as the \mathbb{Z}_2 -orbifold of the symplectic fermion vertex operator superalgebra [1] and ADE-type orbifolds for the triplet vertex algebra $\mathcal{W}(p)$. As we had demonstrated in [4]-[6] (jointly with X.Lin), for the triplet vertex algebra, we have a very strong evidence that all irreducible

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modules for $\mathcal{W}(p)^G$ arise in this way. Let us briefly summarize the main results obtained in those papers. For the A-type (cyclic) orbifolds, based on the analysis of the Zhu algebra, we obtain classification of irreducible modules for $\mathcal{W}(p)^{A_2}$ for small p [4]. After that, in [5], we studied the D -series (dihedral) orbifolds of $\mathcal{W}(p)$. Among many results, we classified irreducible $\mathcal{W}(p)^{D_2}$ -modules again for small p . Finally, in [6] we reduced the classification for all A and D -type orbifolds to a combinatorial problem related to certain constant term identities. However, these constant term identities are very difficult to prove even for moderately small values of p .

In this paper we revisited the classification of modules for the cyclic orbifolds by using a different circle of ideas. Instead of working with the Zhu algebra we employ the singlet vertex subalgebra to classify irreducible $\mathcal{W}(p)^{A_m}$ -modules. Here is the content of the paper with the main results.

In Section 2, we review several basic notions pertaining to singlet, doublet and triplet vertex algebras, including A_n -orbifolds. After that, we prove that the classification problem for $\mathcal{W}(p)^{A_n}$, as conjectured in [4], follows from a certain fusion rules conjecture for the singlet vertex algebra module and simple currents. In Section 3, we study fusion rules for the singlet vertex algebra for $p = 2$. By using Zhu's algebra we obtain an upper bound on the fusion rules of typical modules. In Section 4, which is the most original part of the paper, we obtained a new realization of the doublet (and thus of the singlet and triplet) vertex algebra for $p = 2$ and $p = 3$. We also have a conjectural realization for all $p \geq 4$. Finally, in Section 5, by using the construction in Section 4, we construct all intertwining operators among singlet modules when $p = 2$. Combined with the previously obtained upper bound for generic modules, this proves that some special singlet algebra modules are simple currents, in a suitable sense. Using these results, we prove the completeness of classification of irreducible modules for the orbifold algebra $\mathcal{W}(2)^{A_m}$.

2. CLASSIFICATION OF IRREDUCIBLE MODULES FOR ORBIFOLD $\mathcal{W}(p)^{A_m}$: A FUSION RULES APPROACH

Let $L = \mathbb{Z}\alpha$ be a rank one lattice with $\langle \alpha, \alpha \rangle = 2p$ ($p \geq 1$). Let

$$V_L = \mathcal{U}(\widehat{\mathfrak{h}}_{<0}) \otimes \mathbb{C}[L]$$

denote the corresponding lattice vertex algebra [19], where $\widehat{\mathfrak{h}}$ is the affinization of $\mathfrak{h} = \mathbb{C}\alpha$, and $\mathbb{C}[L]$ is the group algebra of L . Let $M(1)$

be the Heisenberg vertex subalgebra of V_L generated by $\alpha(-1)\mathbf{1}$ with the conformal vector

$$\omega = \frac{\alpha(-1)^2}{4p}\mathbf{1} + \frac{p-1}{2p}\alpha(-2)\mathbf{1}.$$

With this choice of ω , V_L has a vertex operator algebra structure of central charge

$$c_{1,p} = 1 - \frac{6(p-1)^2}{p}.$$

Let $Q := \text{Res}_x Y(e^\alpha, x) = e_0^\alpha$, where we use e^β to denote vectors in the group algebra of the dual lattice of L . The triplet vertex algebra $\mathcal{W}(p)$ is strongly generated by ω and three primary vectors

$$F = e^{-\alpha}, \quad H = QF, \quad E = Q^2F$$

of conformal weight $2p-1$ [8]. We use $H(n) := \text{Res}_x x^{2p-2+n}Y(H, x)$. The doublet vertex algebra $\mathcal{A}(p)$ (see [16, 11]) is a generalized vertex algebra strongly generated by ω and

$$\bar{a}^- = e^{-\alpha/2}, \quad \bar{a}^+ = Qe^{-\alpha/2}.$$

We use $\overline{M(1)}$ to denote the singlet vertex algebra. It is defined as a vertex subalgebra of $\mathcal{W}(p)$ generated by ω and H .

It was proved in [2] that the irreducible \mathbb{N} -graded $\overline{M(1)}$ -modules are parametrized by highest weights with respect to $(L(0), H(0))$ which has the form

$$\lambda_t := \left(\frac{1}{4p}t(t+2p-2), \begin{pmatrix} t \\ 2p-1 \end{pmatrix} \right).$$

The irreducible $\overline{M(1)}$ -module M_t with highest weight λ_t is realized as an irreducible subquotient of the $M(1)$ -module

$$M(1, \frac{t}{2p}\alpha) := M(1) \otimes e^{\frac{t}{2p}\alpha}.$$

For $n \in \mathbb{N}$, we define:

$$\pi_n := M_{-np} = \overline{M(1)} \cdot e^{-n\alpha/2} \quad \text{and} \quad \pi_{-n} := M_{np} = \overline{M(1)} \cdot Q^n e^{-n\alpha/2}.$$

We know [8] that π_j , $j \in \mathbb{Z}$, are irreducible as $\overline{M(1)}$ -modules and $\pi_j \subset M(1, -j\alpha/2) = M(1) \otimes e^{-j\alpha/2}$.

Recall that the vertex algebra $\mathcal{W}(p)^{A_m}$ is a \mathbb{Z}_m -orbifold of $\mathcal{W}(p)$ and is generated by

$$\omega, F^{(m)} = e^{-m\alpha}, \quad E^{(m)} = Q^{2m}F^{(m)}, \quad H = Qe^{-\alpha};$$

for details see [4]. Therefore $\mathcal{W}(p)^{A_m}$ is an extension of the singlet algebra and we have:

$$\mathcal{W}(p)^{A_m} = \bigoplus_{n \in \mathbb{Z}} \pi_{2nm}.$$

Theorem 2.1. [4] *For every $0 \leq i \leq 2pm^2 - 1$, there exists a unique (up to equivalence) irreducible $\mathcal{W}(p)^{A_m}$ -module $L_{\frac{i}{m}}$ which decomposes as the following direct sum of $\overline{M(1)}$ -modules*

$$L_{\frac{i}{m}} = \bigoplus_{n \in \mathbb{Z}} M_{\frac{i}{m} - 2pmn}.$$

Moreover, $L_{\frac{i}{m}}$ is realized as an irreducible subquotient of $V_{\frac{i}{2pm}\alpha + \mathbb{Z}(m\alpha)}$.

Definition 2.2. *We say that an irreducible \mathbb{N} -graded $\overline{M(1)}$ -module W_1 is a simple-current in the category \mathbb{N} -graded $\overline{M(1)}$ -modules if for every irreducible \mathbb{N} -graded module W_2 , there is a unique irreducible \mathbb{N} -graded $\overline{M(1)}$ -module W_3 such that the vector space of intertwining operators $I_{\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)}$ is 1-dimensional and $I_{\left(\begin{smallmatrix} N \\ W_1 \ W_2 \end{smallmatrix}\right)} = 0$ for any other irreducible \mathbb{N} -graded $\overline{M(1)}$ -module which is not isomorphic to W_3 . We write*

$$W_1 \times W_2 = W_3.$$

Note that we require simple current property to hold only on irreducible $\overline{M(1)}$ -modules; studying fusion products (= tensor products) with indecomposable modules is a much harder problem which we do not address here.

Conjecture 2.3. *Let $p \in \mathbb{Z}_{\geq 2}$. Modules π_j , $j \in \mathbb{Z}$ are simple currents in the category of ordinary, \mathbb{N} -graded $\overline{M(1)}$ -modules, and the following fusion rules holds:*

$$\pi_j \times M_t = M_{t-jp}.$$

This conjecture is in agreement with the fusion rules result from [12] based on the Verlinde formula of characters. In Section 3, we shall prove this conjecture in the case $p = 2$.

Lemma 2.4. *Assume that Conjecture 2.3 holds. Assume that M is an irreducible $\mathcal{W}(p)^{A_m}$ -module generated by a highest weight vector v_t for $\overline{M(1)}$ of the $(L(0), H(0))$ -weight λ_t . Then $t \in \frac{1}{m}\mathbb{Z}$.*

Proof. By Conjecture 2.3, we see that π_{2m} sends $M_t := \overline{M(1)}v_t$ to M_{t-2mp} , where $M_{t-2mp} = \overline{M(1)}v_{t-2mp}$ is an irreducible $\overline{M(1)}$ -module

of highest weight

$$\left(\frac{1}{4p}(t - 2mp)(t - 2mp - 2p + 2), \binom{t - 2mp}{2p - 1} \right).$$

But the difference between conformal weights of v_t and v_{t-2mp} must be an integer. This leads to the following condition

$$m(t - p + 1 + mp) = \frac{1}{4p}(t + 2mp)(t + 2mp - 2p + 2) - \frac{1}{4p}t(t - 2p + 2) \in \mathbb{Z}.$$

Thus, $mt \in \mathbb{Z}$. \square

Theorem 2.5. *Assume that Conjecture 2.3 holds. Then the vertex operator algebra $\mathcal{W}(p)^{A_m}$ has precisely $2m^2p$ -irreducible modules, all constructed in [4].*

Proof. Clearly, any irreducible $\mathcal{W}(p)^{A_m}$ -module M is non-logarithmic (i.e., $L(0)$ -acts semisimple). Such module has weights bounded from below, so M must contain a singular vector v_t for $\overline{M(1)}$. Then by using Lemma 2.4 we get that as a $\overline{M(1)}$ -module,

$$M \cong \bigoplus_{n \in \mathbb{Z}} M_{t-2mp}$$

where $t \in \frac{1}{m}\mathbb{Z}$ and M_{t-2mp} is the irreducible highest weight $\overline{M(1)}$ -module with highest weight

$$\lambda_{t-2mp} = \left(\frac{1}{4p}(t - 2mp)(t - 2mp - 2p + 2), \binom{t - 2mp}{2p - 1} \right).$$

Now, the assertion follows from Theorem 2.1. \square

Remark 2.6. *A different approach to the problem of classification of irreducible $\mathcal{W}(p)^{A_m}$ -modules was presented in [6]. It is based on certain constant term identities and it does not require any knowledge of fusion rules among $\overline{M(1)}$ -modules.*

3. FUSION RULES FOR $\overline{M(1)}$ AND A PROOF OF CONJECTURE 2.3 FOR $p = 2$.

Let V be a vertex operator algebra, W_1, W_2, W_3 be V -modules, and \mathcal{Y} be an intertwining operator of type

$$\begin{pmatrix} W_3 \\ W_1 \ W_2 \end{pmatrix}.$$

For $u \in W_1$, we use $\mathcal{Y}(u, z) = \sum_{r \in \mathbb{C}} u_r z^{-r-1}$, $u_r \in \text{Hom}(W_2, W_3)$.

Let U_i be a V -submodule of W_i , $i = 1, 2$. Define the inner *fusion product*

$$(1) \quad U_1 \cdot U_2 = \text{span}_{\mathbb{C}} \{u_r v \mid u \in U_1, v \in U_2, r \in \mathbb{C}\}.$$

Then $U_1 \cdot U_2$ is a V -submodule of W_3 .

Let $M(1, \lambda)$ be as before. Then $M(1, \lambda)$ is an $\overline{M(1)}$ -module and e^λ of highest weight

$$\left(\frac{1}{4p} t(t - 2p + 2), \binom{t}{2p - 1} \right), \quad t = \langle \lambda, \alpha \rangle.$$

We specialize now $p = 2$. For $t \notin \mathbb{Z}$, the irreducible $\overline{M(1)}$ -module M_t remains irreducible as a Virasoro module [17] and it can be realized as $M(1, \lambda)$.

Denote by $A(\overline{M(1)})$ the Zhu associative algebra of $\overline{M(1)}$. For a $\overline{M(1)}$ -module M , denote by $A(M)$ the $A(\overline{M(1)})$ -bimodule with the left and right action given by

$$\begin{aligned} a * m &= \text{Res}_x \frac{(1+x)^{\deg(a)}}{x} Y(a, x) m, \\ m * a &= \text{Res}_x \frac{(1+x)^{\deg(a)-1}}{x} Y(a, x) m, \end{aligned}$$

respectively. It is known that there is an injective map from the space of intertwining operators of type $\binom{M_3}{M_1 \ M_2}$ to

$$\text{Hom}_{A(V)}(A(M_1) \otimes_{A(V)} M_2(0), M_3(0))$$

where $M_i(0)$ is the top degree of M_i . Next, we analyze the space

$$A(M_1) \otimes_{A(V)} M_2(0)$$

in the case when $V = \overline{M(1)}$, $M_1 = M(1, \lambda)$ and $M_2 = M(1, \mu)$. We denote by $L^{Vir}(c, 0)$ the simple Virasoro vertex algebra of central charge c . Clearly, $M(1, \lambda)$ can be viewed as a Virasoro vertex algebra module. As such, we have $A_{Vir}(M(1, \lambda)) \cong \mathbb{C}[x, y]$ as an $A(L^{Vir}(c, 0))$ -bimodule. Therefore, $A(M(1, \lambda)) \cong \mathbb{C}[x, y]/I$, where I is the ideal coming from extra relations in $\overline{M(1)}$.

Observe also that inside $A(M(1, \lambda)) \otimes_{A(\overline{M(1)})} \mathbb{C}_\mu$, where $\mathbb{C}_\mu = \mathbb{C}v_\mu$ is one-dimensional $A(\overline{M(1)})$ -module spanned by the highest weight

vector v_μ , we have the following relations (here $v \in A(M(1, \lambda))$):

$$\begin{aligned} v * [H] \otimes v_\lambda - \binom{\langle \mu, \alpha \rangle}{3} v \otimes v_\mu &= 0, \\ v * [\omega] \otimes v_\lambda - \frac{\langle \mu, \alpha \rangle (\langle \mu, \alpha \rangle - 2)}{8} v \otimes v_\mu &= 0, \\ \text{Res}_x \frac{(1+x)^3}{x^2} Y(H, x) v &= (H_{-2} + 3H_{-1} + 3H_0 + H_1) v \in O(M(1, \lambda)). \end{aligned}$$

Let

$$W(\lambda, \mu) := A(M(1, \lambda)) \otimes_{A(M(1))} \mathbb{C}_\mu \cong \mathbb{C}[x]/J.$$

Then we have

Theorem 3.1. *Suppose $t = \langle \alpha, \lambda \rangle \notin \mathbb{Z}$, and $s = \langle \alpha, \mu \rangle \notin \mathbb{Z}$. Then*

$$p(x) = \left(x - \frac{(s+t)(s+t-2)}{8}\right) \left(x - \frac{(s+t-2)(s+t-4)}{8}\right) \in J.$$

In particular, $W(\lambda, \mu)$ is at most 2-dimensional.

Proof. Irreducibility of $M(1, \lambda)$ with respect to Virasoro algebra, implies that $H_i v_\lambda, i \geq -2$ can be written in the Virasoro basis. We omit these explicit formulas here obtained by a straightforward tedious computations. Together with

$$(H_{-2} + 3H_{-1} + 3H_0 + H_1) v_\lambda = 0,$$

we obtain a new relation $f(x, y) = 0$ inside $A(M(1, \lambda))$. This relation turns into

$$\left(x - \frac{(t+s)(t+s-2)}{8}\right) \left(x - \frac{(t+s-2)(t+s-4)}{8}\right) \left(x - \frac{(s-t)(s-t-2)}{8}\right) \left(x - \frac{(s-t+2)(s-t)}{8}\right) = 0$$

inside $W(\lambda, \mu)$. Similarly,

$$\begin{aligned} 0 &= v_\lambda * [H] \otimes v_\mu - \binom{s}{3} v \otimes v_\mu \\ &= (H_{-1} + 2H_0 + H_1 - \binom{s}{3}) v_\lambda \otimes v_\mu \end{aligned}$$

turns into

$$\left(x - \frac{(t+s)(t+s-2)}{8}\right) \left(x - \frac{(t+s-2)(t+s-4)}{8}\right) \left(x - \left(\frac{s^2}{8} + \frac{t^2}{8} - \frac{st}{2} + \frac{s}{4} + \frac{t}{4}\right)\right) = 0$$

inside $W(\lambda, \mu) = 0$. Now we have to analyze common roots of these two polynomials. Clearly, they have $\left(x - \frac{(t+s)(t+s-2)}{8}\right) \left(x - \frac{(t+s-2)(t+s-4)}{8}\right)$ in common. It remains to analyse whether

$$\frac{(s-t)(s-t-2)}{8} = \frac{s^2}{8} + \frac{t^2}{8} - \frac{st}{2} + \frac{s}{4} + \frac{t}{4}$$

or

$$\frac{(s-t)(s-t+2)}{8} = \frac{s^2}{8} + \frac{t^2}{8} - \frac{st}{2} + \frac{s}{4} + \frac{t}{4}$$

The first equation holds if $t = 2$ (for all s) and also for $s = 0$ (for all t). The second equation holds for $s = 2$ (for all t) and also for $t = 0$ (for all s). But this means either s or t are integral. The initial assumptions on s and t now yield the claim. \square

Remark 3.2. *By using exactly the same method we can prove that for $p = 3$ the space $W(\lambda, \mu)$ is at most 3-dimensional, as predicted in [12].*

We shall use the following result on intertwining operators for the Virasoro algebra.

Lemma 3.3. *Assume that there is a non-trivial intertwining operator $I(\cdot, z)$ of type*

$$\begin{pmatrix} M(1, \nu) \\ L^{Vir}(-2, (n^2 + n)/2) \quad M(1, \mu) \end{pmatrix}$$

such that $I(v^{(n)}, z) = \sum_{r \in \mathbb{C}} v_r^{(n)} z^{-r-1}$ and there is $r_0 \in \mathbb{C}$ such that

$$(2) \quad v_{r_0}^{(n)} v_\mu = \lambda v_\nu, \quad (\lambda \neq 0).$$

(Here $v^{(n)}$ is highest weight vector in $L^{Vir}(-2, (n^2 + n)/2)$). Then

$$\nu = \mu + (i - n/2)\alpha, \quad \text{for } 0 \leq i \leq n.$$

Proof. This is essentially proven in Theorem 5.1 in [20]. \square

Theorem 3.4. *Let $p = 2$. Modules π_j , $j \in \mathbb{Z}$ are simple currents in the category of ordinary, \mathbb{N} -graded $\overline{M(1)}$ -modules.*

Proof. Let us here consider the case $j > 0$ (the case $j < 0$ can be treated similarly). Then $t = -2j \notin \{0, 2\}$. Let as above M_s and M_r be irreducible $\overline{M(1)}$ -modules with highest weight λ_s and λ_r , respectively. By using the proof of Theorem 3.1 we see that if

$$I \begin{pmatrix} M_r \\ \pi_j \quad M_s \end{pmatrix} \neq 0,$$

then exactly one of the following holds:

- (1) $s \notin \{0, 2\}$ and $r \in \{s - 2j, s - 2j - 2\}$,
- (2) $s = 0$, $r \in \{2j, 2j + 2, 2j + 4, -2j - 2, -2j, -2j + 2\}$,
- (3) $s = 2$, $r \in \{2j, 2j + 2, 2j + 4, -2 - 2j, -2j, -2j + 2\}$.

Cases (2) and (3) come from analyzing additional solutions. By using the following relation in $A(M(1, \lambda))$

$$H * v_\lambda - v_\lambda * H = (H_0 + 2H_1 + H_2)v_\lambda$$

we get that

$$r = s+t \quad \text{or} \quad r = s+t-2 \quad \text{or} \quad r = 1/3(5-3s+t \pm \sqrt{1+12s+4t-12st+4t^2}).$$

This implies

- (1') $s \notin \{0, 2\}$ and $r \in \{s-2j, s-2j-2\}$,
- (2') $s = 0, r \in \{-2j-2, -2j, -2j+2, \frac{4+2j}{3}\}$,
- (3') $s = 2, r \in \{-2j-2, -2j, -2j+2, \frac{4+2j}{3}\}$.

On the other hand since $L^{Vir}(-2, (j^2+j)/2)$ is an irreducible Virasoro submodule of π_j , then in the category of $L^{Vir}(-2, 0)$ -modules we have a non-trivial intertwining operator of the type

$$\begin{pmatrix} M_r \\ L^{Vir}(-2, (j^2+j)/2) & M_s \end{pmatrix}.$$

Now, applying Virasoro fusion rules computed in Lemma 3.3 we get

$$r \in \{s-2j, s-2j+4, \dots, s+2j-4, s+2j\}.$$

By analyzing all cases above, we see that the only possibility is $r = s-2j$. The proof follows. \square

4. DEFORMED REALIZATION OF THE TRIPLET AND SINGLET VERTEX ALGEBRA

4.1. Definition of deformed realization. In this section we present a new realization of the triple and doublet vertex operator algebra. This is then used to show that the sufficient conditions obtained in Theorem 3.1 are in fact necessary. Let $p \in \mathbb{Z}_{>0}$. Let V_L be the lattice vertex algebra associated to the lattice

$$L = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2, \quad \langle \alpha_1, \alpha_1 \rangle = 2p-1, \quad \langle \alpha_2, \alpha_2 \rangle = 1, \quad \langle \alpha_1, \alpha_2 \rangle = 0.$$

Let $\alpha = \alpha_1 + \alpha_2$. Define

$$\omega = \frac{1}{4p}\alpha(-1)^2 + \frac{p-1}{2p}\alpha(-2) + \frac{p-1}{p}e^{-2\alpha_2}.$$

Then ω generates the Virasoro vertex operator algebra $L(c_{1,p}, 0)$ of central charge $c_{1,p} = 1 - 6(p-1)^2/p$. More generally,

Proposition 4.1. *Let $\omega' \in V$ be a conformal vector of central charge c and $v \in V_1$ (degree one) a primary vector for ω' such that $[v_n, v_m] = 0$ for all $m, n \in \mathbb{Z}$. Then*

$$\omega = \omega' + v$$

is a conformal vector of the same central charge.

Consider the following operator

$$S = e_0^{\alpha_1 + \alpha_2} + e_0^{\alpha_1 - \alpha_2}$$

acting on V_L .

Lemma 4.2. *The operator S is a screening operator for the Virasoro algebra generated by ω .*

Proof. We have to prove that

$$[e_0^{\alpha_1 + \alpha_2} + e_0^{\alpha_1 - \alpha_2}, L'(n) + \frac{p-1}{p}e_{n+1}^{-2\alpha_2}] = 0.$$

This follows directly from the relations

$$\begin{aligned} [e_0^{\alpha_1 + \alpha_2}, L'(n)] &= 0, \quad L'(0)e^{\alpha_1 - \alpha_2} = 0 \\ [e_0^{\alpha_1 - \alpha_2}, e_{n+1}^{-2\alpha_2}] &= 0, \quad [e_0^{\alpha_1 + \alpha_2}, e_{n+1}^{-2\alpha_2}] = ((\alpha_1(-1) + \alpha_2(-1))e^{\alpha_1 - \alpha_2})_{n+1} \\ [L(n)', e_0^{\alpha_1 - \alpha_2}] &= (L(-1)'e^{\alpha_1 - \alpha_2})_{n+1} = \frac{p-1}{p}((\alpha_1(-1) + \alpha_2(-1)))e^{\alpha_1 - \alpha_2}_{n+1}. \end{aligned}$$

□

Let $\widetilde{\mathcal{W}}(p)$ be the vertex subalgebra of V_L generated by

$$\omega, F = e^{-\alpha}, H = SF, E = S^2F.$$

Let $\widetilde{\mathcal{A}}(p)$ be the (generalized) vertex algebra generated by

$$\begin{aligned} a^- &= e^{-\alpha/2}, \\ a^+ &= Sa^- = S_{p-1}(\alpha)e^{\alpha/2} + S_{p-2}(\alpha_1 - \alpha_2)e^{\alpha/2 - 2\alpha_2}. \end{aligned}$$

Clearly, $\widetilde{\mathcal{W}}(p) \subset \widetilde{\mathcal{A}}(p)$. Let $\widetilde{\mathcal{W}}^0(p)$ be the vertex subalgebra of $\widetilde{\mathcal{W}}(p)$ generated by ω and

$$H = SF = S_{2p-1}(\alpha) + S_{2p-3}(\alpha_1 - \alpha_2)e^{-2\alpha_2}.$$

Conjecture 4.3. *We have*

$$\widetilde{\mathcal{A}}(p) \cong \mathcal{A}(p), \quad \widetilde{\mathcal{W}}(p) \cong \mathcal{W}(p), \quad \widetilde{\mathcal{W}}^0(p) \cong \overline{M(1)},$$

as vertex algebras.

Theorem 4.4. *Conjecture 4.3 holds for $p = 2$ and $p = 3$.*

Proof. Let

$$\begin{aligned} a^- &= e^{-\alpha/2}, \\ a^+ &= Sa^- = S_{p-1}(\alpha)e^{\alpha/2} + S_{p-2}(\alpha_1 - \alpha_2)e^{\alpha/2 - 2\alpha_2}. \end{aligned}$$

Case $p = 2$.

Then

$$a^+ = \alpha(-1)e^{\alpha/2} + e^{\alpha/2-2\alpha_2}.$$

Direct calculation shows that

$$a_n^- a^+ = 2\delta_{n,1}\mathbf{1}, \quad a_n^- a^+ = 0 \quad (n \geq 0),$$

and by (super)commutator formulae we have

$$(3) \quad \{a_n^-, a_m^+\} = 2n\delta_{n+m,0}, \quad \{a_n^\pm, a_m^\pm\} = 0 \quad (n, m \in \mathbb{Z}).$$

This implies a^- and a^+ generate the subalgebra isomorphic to the symplectic fermion vertex superalgebra $\mathcal{A}(2)$ (cf. [1], [18], [21]). Since even part of the symplectic fermions is isomorphic to the triplet vertex algebra $\mathcal{W}(p)$ for $p = 2$, the assertion holds.

Case $p = 3$. Let $F_{-p/2}$ be the generalized lattice vertex algebra associated to the lattice $L = \mathbb{Z}\varphi$, $\langle \varphi, \varphi \rangle = -p/2$ with generators $e^{\pm\varphi}$ (cf. [15]) and $V^{-4/3}(sl_2)$ the universal affine vertex algebra at the admissible level $-\frac{4}{3}$. As in [3], one shows that for $p = 3$

$$(4) \quad e = a^- \otimes e^{-\varphi}, \quad f = -\frac{2}{9}a^+ \otimes e^\varphi, \quad h = \frac{4}{3}\varphi$$

defines a non-trivial homomorphism

$$\Phi : V^{-4/3}(sl_2) \rightarrow \tilde{\mathcal{A}}(3) \otimes F_{-3/2}$$

which maps singular vector in $V^{-4/3}(sl_2)$ to zero (calculation is completely analogous as in [3]). Therefore (4) gives an embedding of the simple affine vertex algebra $V_{-4/3}(sl_2)$ into $\tilde{\mathcal{A}}(3) \otimes F_{-3/2}$. By construction, we have that the vacuum space

$$\Omega(V_{-4/3}(sl_2)) = \{v \in V_{-4/3}(sl_2) \mid h(n)v = 0 \ \forall n \geq 1\}$$

is generated by a^+ and a^- and therefore

$$\Omega(V_{-4/3}(sl_2)) \cong \tilde{\mathcal{A}}(3).$$

In [3], it was proven that

- The vacuum space $\Omega(V_{-4/3}(sl_2))$ is isomorphic to the doublet vertex algebra $\mathcal{A}(3)$.
- The parafermionic vertex algebra

$$K(sl_2, -4/3) = \{v \in V_{-4/3}(sl_2) \mid h(n)v = 0 \ \forall n \geq 0\}$$

is isomorphic to $\mathcal{W}^0(3) \cong \overline{M(1)}$.

So we have

$$\begin{aligned} \tilde{\mathcal{A}}(3) &\cong \Omega(V_{-4/3}(sl_2)) \cong \mathcal{A}(3), \\ \widetilde{\mathcal{W}}^0(3) &\cong K(sl_2, -4/3) \cong \mathcal{W}^0(3). \end{aligned}$$

Since the triplet $\mathcal{W}(3)$ is a \mathbb{Z}_2 -orbifold of $\mathcal{A}(3)$, we also have that $\widetilde{\mathcal{W}}(3) \cong \mathcal{W}(3)$. This proves the conjecture in the case $p = 3$. \square

4.2. On the proof of Conjecture 4.3 for $p \geq 4$. At this point, we do not have yet a uniform proof of Conjecture 4.3 for all p . The problem is that $\mathcal{W}(p)$ and $\overline{M(1)}$ are not members of a family of generic vertex algebras and \mathcal{W} -algebras. In order to solve this difficulty, one can study the embeddings of $\mathcal{W}(p)$ and $\overline{M(1)}$ into affine vertex algebras and minimal \mathcal{W} -algebras (cf. [3], [13]). The case $p = 3$ was already discussed in the proof of Theorem 4.4. A similar arguments can be repeated in the $p = 4$ case. For $p = 4$ one can show that

$$(5) \quad a^+ \otimes e^\varphi, \quad a^- \otimes e^{-\varphi}$$

generate the simple vertex algebra $\mathcal{W}_3^{(2)}$ with central charge $-23/2$ whose vacuum space is generated by a^+ and a^- . By using the realization from [13], we get that a^+ and a^- generate the doublet vertex algebra $\mathcal{A}(4)$. So Conjecture 4.3 also holds for $p = 4$. For $p \geq 5$, one needs to prove that (5) define the Feigin-Semikhatov W -algebra $W_p^{(2)}$ embedded into $\mathcal{A}(p) \otimes F_{-p/2}$. But this calculation is much more complicated.

4.3. Deformed action for $p = 2$. Let now $p = 2$. Then the singlet vertex algebra $\overline{M(1)}$ is isomorphic to a subalgebra of V_L generated by

$$\begin{aligned} \omega &= \frac{1}{8}\alpha(-1)^2 + \frac{1}{4}\alpha(-2) + \frac{1}{2}e^{-2\alpha_2}, \\ H &= Se^{-\alpha} = S_3(\alpha) + (\alpha_1(-1) - \alpha_2(-2))e^{-2\alpha_2}. \end{aligned}$$

Let $r, s \in \mathbb{C}$. Consider the $\overline{M(1)}$ -module

$$\mathcal{F}(r, s) := \overline{M(1)}.v_{r,s}, \quad v_{r,s} = e^{\frac{r}{3}\alpha_1 + s\alpha_2}.$$

We have:

$$\begin{aligned}
L(0)v_{r,0} &= h_{r+1,1}v_{r,0}, \\
L(0)v_{r,-1} &= h_{r,1}v_{r,-1} \\
L(0)v_{r,1} &= h_{r+2,1}v_{r,1} + \frac{1}{2}v_{r,-1}, \\
L(0)(v_{r,1} - \frac{1}{2h_{r,1}}v_{r,-1}) &= h_{r+2,1}v_{r,1}, \\
H(0)v_{r,0} &= \binom{r}{3}v_{r,0}, \\
H(0)v_{r,-1} &= \binom{r-1}{3}v_{r,-1}, \\
H(0)v_{r,1} &= \binom{r+1}{3}v_{r,1} + (r-1)v_{r,-1},
\end{aligned}$$

where

$$h_{r,1} = \frac{(r-1)^2}{8} - \frac{(r-1)}{4} = \frac{(r-2)^2 - 1}{8}.$$

Proposition 4.5.

(1) Assume that $r \notin \mathbb{Z}$. Then

$$\mathcal{P}_r := \overline{M(1)}.v_{r,1} = \overline{M(1)}.v_{r,1}^+ \oplus \overline{M(1)}.v_{r,-1}^-$$

where

$$v_{r,1}^+ = v_{r,1} + \frac{p-1}{p} \frac{1}{h_{r+2,1} - h_{r,1}} v_{r,-1} = v_{r,1} + \frac{1}{r-1} v_{r,-1}, \quad v_{r,1}^- = v_{r,-1}$$

are highest weight vectors for $\overline{M(1)}$ of $(L(0), H(0))$ weight

$$\left(h_{r+2,1}, \binom{r+1}{3} \right), \quad \left(h_{r,1}, \binom{r-1}{3} \right)$$

respectively. In particular,

$$\mathcal{P}_r = M_{r+1} \oplus M_{r-1}.$$

(2) Assume that $r = p-1 = 1$. Then $\mathcal{P}_1 := \overline{M(1)}.v_{1,1}$ is a $\mathbb{Z}_{\geq 0}$ -graded logarithmic $\overline{M(1)}$ -module. The lowest component $\mathcal{P}_1(0)$ is 2-dimensional, spanned by $v_{1,1}$ and $v_{1,-1}$. For $r \in \mathbb{Z} \setminus \{0\}$, see also Remark 5.2.

Proof. Parts (1) and (2) follow immediately due to irreducibility of M_t and because $L(0)$ forms a Jordan block on the top component, respectively. \square

Remark 4.6. In [10], the authors presented an explicit realization of certain logarithmic modules for $\mathcal{W}(p)$. Our deformed realization of $\mathcal{W}(p)$ gives an alternative realization of these logarithmic modules. We plan to return to these realizations in our forthcoming publications.

5. INTERTWINING OPERATORS BETWEEN TYPICAL $\overline{M(1)}$ AND $\mathcal{W}(p)^{A_m}$ -MODULES: THE $p = 2$ CASE

In this section we fix $p = 2$.

Theorem 5.1.

(1) Assume that $t_1, t_2, t_1 + t_2 \notin \mathbb{Z}$. Then in the category of $\overline{M(1)}$ -modules, there exists non-trivial intertwining operators of types

$$\begin{pmatrix} M_{t_1+t_2} \\ M_{t_1} & M_{t_2} \end{pmatrix}, \quad \begin{pmatrix} M_{t_1+t_2-2} \\ M_{t_1} & M_{t_2} \end{pmatrix}.$$

In particular, the following fusion rules holds:

$$M_{t_1} \times M_{t_2} = M_{t_1+t_2} + M_{t_1+t_2-2}.$$

Proof. In our realization, $\overline{M(1)}$ is a vertex subalgebra of the vertex algebra $U := M_{\alpha_1}(1) \otimes V_{\mathbb{Z}2\alpha_2}$, where $M_{\alpha_1}(1)$ is the Heisenberg vertex algebra, generated by α_1 , and $V_{\mathbb{Z}2\alpha_2}$ is the lattice vertex algebra associated to the even lattice $\mathbb{Z}(2\alpha_1)$. First we notice that in the category of U -modules we have a non-trivial intertwining operator \mathcal{Y} of type

$$\begin{pmatrix} M_{\alpha_1}(1, \frac{t_1+t_2-1}{3}\alpha_1) \otimes V_{\alpha_2+\mathbb{Z}2\alpha_2} \\ M_{\alpha_1}(1, \frac{t_1}{3}\alpha_1) \otimes V_{\mathbb{Z}(2\alpha_2)} & M_{\alpha_1}(1, \frac{t_2-1}{3}\alpha_1) \otimes V_{\alpha_2+\mathbb{Z}(2\alpha_2)} \end{pmatrix}.$$

Clearly,

$$\begin{aligned} M_{t_1} &= \overline{M(1)}.v_{t_1,0} \subset M_{\alpha_1}(1, \frac{t_1}{3}\alpha_1) \otimes V_{\mathbb{Z}(2\alpha_2)} \\ M_{t_2} &= \overline{M(1)}.v_{t_2-1,1}^+ \subset M_{\alpha_1}(1, \frac{t_2-1}{3}\alpha_1) \otimes V_{\alpha_2+\mathbb{Z}2\alpha_2} \\ M_{t_1+t_2} &= \overline{M(1)}.v_{t_1+t_2-1,1}^+ \subset M_{\alpha_1}(1, \frac{t_1+t_2-1}{3}\alpha_1) \otimes V_{\alpha_2+\mathbb{Z}2\alpha_2} \\ M_{t_1+t_2-2} &= \overline{M(1)}.v_{t_1+t_2-1,1}^- \subset M_{\alpha_1}(1, \frac{t_1+t_2-1}{3}\alpha_1) \otimes V_{\alpha_2+\mathbb{Z}2\alpha_2} \end{aligned}$$

By abusing the notation, let \mathcal{Y} denotes the restriction to M_{t_1} as defined above. We have to first check that \mathcal{Y} satisfies the $L(-1)$ -property for the new Virasoro generator $L(-1) = L'(-1) + \frac{1}{2}e_0^{-2\alpha_2}$. Observe that $\mathcal{Y}(L(-1)v_{t_1,0}) = \mathcal{Y}(L'(-1)v_{t_1,0}, x) = \frac{d}{dx}\mathcal{Y}(v_{t_1,0}, x)$. For other vectors, the $L(-1)$ property follows from the fact that $M_{t,1}$ is a cyclic module.

By using Theorem 3.1 and the fact that $M_{t_1} \cdot M_{t_2}$ is completely reducible, we see that the corresponding fusion product (1), defined via the intertwining operator \mathcal{Y} , must satisfy $M_{t_1} \cdot M_{t_2} \subset M_{t_1+t_2} \oplus M_{t_1+t_2-2}$. By construction, we have that

$$\begin{aligned} w &= v_{t_1+t_2-1,1} + \frac{1}{t_2-2} v_{t_1+t_2-1,-1} \\ &= v_{t_1+t_2-1,1}^+ + \frac{t_1}{(t_2-2)(t_1+t_2-2)} v_{t_1+t_2-1,1}^- \in M_{t_1} \cdot M_{t_2}, \end{aligned}$$

which implies that singular vectors $v_{t_1+t_2-1,1}^\pm$ belong to $M_{t_1} \cdot M_{t_2}$. So we get that $M_{t_1} \cdot M_{t_2} = M_{t_1+t_2} \oplus M_{t_1+t_2-2}$ as desired. \square

Remark 5.2. One can show that for $t_1, t_2 \notin \mathbb{Z}$, $t_1 + t_2 \in \mathbb{Z}$, our free field realizations in Proposition 4.5 also gives the existence of a logarithmic intertwining operator of type

$$\begin{pmatrix} \mathcal{P}_{t_1+t_2} \\ M_{t_1} & M_{t_1} \end{pmatrix}.$$

Moreover, observe that the condition $t_1, t_2 \notin \mathbb{Z}$ is not necessary here and we may assume $t_1, t_2 \in \{4n+1, 4n-1 : n \in \mathbb{Z}\}$. This leads to a new family of intertwining operator presumably obtained by the restriction of an intertwining operator coming from a triple of symplectic fermions modules. We shall study these intertwining operators in more details in our future publications.

We have similar result in the category of $\mathcal{W}(p)^{A_m}$ -modules:

Theorem 5.3. Assume that $i_1, i_2, i_3 \in \{0, \dots, 2pm^2 - 1\}$, $i_1, i_2, i_1 + i_2 \notin m\mathbb{Z}$, where m is a positive integer. Then in the category of $\mathcal{W}(p)^{A_m}$ -modules, there exists non-trivial intertwining operators of type

$$\begin{pmatrix} L_{i_3/m} \\ L_{i_1/m} & L_{i_2/m} \end{pmatrix}$$

if and only if

$$i_3 \equiv i_1 + i_2 \pmod{2m^2p} \quad \text{or} \quad i_3 \equiv i_1 + i_2 - 2m \pmod{2m^2p}.$$

Remark 5.4. Results in this paper, together with Remark 5.2, provides a rigorous proof of the main conjecture in [12] pertaining to fusion rules of $\overline{M}(1)$ for $p = 2$.

Almost all results in the paper can be extended for the $N = 1$ singlet/triplet vertex algebras introduced and studied by the authors [9] (work in progress).

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